

Spectral Factorization of Rank-Deficient Polynomial Matrix-Functions

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Abstract. A spectral factorization theorem is proved for polynomial rank-deficient matrix-functions. The theorem is used to construct paraunitary matrix-functions with first rows given.

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Wiener's spectral factorization theorem [12], [4] for polynomial matrix-functions asserts that if

$$(1) \quad S(z) = \sum_{n=-N}^N C_n z^n$$

is an $m \times m$ matrix-function ($C_n \in \mathbb{C}^{m \times m}$ are matrix coefficients) which is positive definite for a.a. $z \in \mathbb{T}$, $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, then it admits a factorization

$$(2) \quad S(z) = S^+(z)S^-(z) = \sum_{n=0}^N A_n z^n \cdot \sum_{n=0}^N A_n^* z^{-n}, \quad z \in \mathbb{C} \setminus \{0\},$$

where S^+ is an $m \times m$ polynomial matrix-function which is nonsingular inside \mathbb{T} , $\det S^+(z) \neq 0$ when $|z| < 1$, and S^- is its adjoint, $A_n^* = \overline{A_n}^T$, $n = 0, 1, \dots, N$. (Respectively, S^- is analytic and nonsingular outside \mathbb{T} .) S^+ is unique up to a constant right unitary multiplier.

The factorization (2) is also known under the name of matrix-valued Fejér-Riesz theorem and its simple proof is provided in [2]. Various practical applications of this theorem in system analysis [6] and wavelet design [1] are widely recognized.

In the present paper we consider rank-deficient matrix polynomials and prove the corresponding spectral factorization theorem for them:

Theorem 1. *Let $S(z)$ be an $m \times m$ (trigonometric) polynomial matrix-function (1) of order N ($C_N = C_{-N}^* \neq \mathbf{0}$) which is nonnegative definite and of rank $k \leq m$ for a.a. $z \in \mathbb{T}$. Then there exists a unique (up to a $k \times k$ unitary matrix right multiplier) $m \times k$ matrix-polynomial $S^+(z) = \sum_{n=0}^N A_n z^n$, $A_n \in \mathbb{C}^{m \times k}$ of order N ($A_N \neq \mathbf{0}$), which is of full rank k for each z inside \mathbb{T} , such that (2) holds.*

Remark. If we require of S^+ to be just a rational matrix-function analytic inside \mathbb{T} and drop the uniqueness from the condition, then the theorem can be obtained in a standard algebraic manner (see [9]). Hence, as we will see below, the proof of the theorem provides a simple proof of the same theorem for the full rank case, $k = m$, as well. This proof is even more elementary as compared with the one given in [2] since it avoids an application of the Hardy space theory.

Prior to proving the theorem, we make some simple observations on adjoint functions and prove Lemma 1 on paraunitary matrix-functions. We do not claim that this lemma is new, but include its proof for the sake of completeness.

If f is an analytic $m \times k$ matrix-function in $\mathbb{C} \setminus \{z_1, z_2, \dots, z_n\}$, then its adjoint $f^*(z) = \overline{f(1/\bar{z})}^T$ is an analytic $k \times m$ matrix-function in $\mathbb{C} \setminus \{z_1^*, z_2^*, \dots, z_n^*\}$, $z^* := 1/\bar{z}$, $\infty^* = 0$. Obviously, if f is analytic inside \mathbb{T} , then f^* is analytic outside \mathbb{T} (including infinity). Namely, if $f_{ij}(e^{i\theta}) \in L_1^+(\mathbb{T})$, (f_{ij} is the ij th entry of f), then $f_{ji}^*(e^{i\theta}) \in L_1^-(\mathbb{T})$, where $L_1^+(\mathbb{T})$ ($L_1^-(\mathbb{T})$) is the set of integrable functions defined on \mathbb{T} which have Fourier coefficients with negative (positive) indices equal to zero. Since f is uniquely determined by its values on \mathbb{T} , and $f^*(z) = \overline{f(z)}^T = (f(z))^*$ for $|z| = 1$, usual relations for adjoint matrix-functions, like $(fg)^*(z) = g^*(z)f^*(z)$ and $(f^{-1})^*(z) = (f^*)^{-1}(z)$, etc., are valid.

Note that if f is a rational $m \times m$ matrix-function, $f \in \mathcal{R}^{m \times m}$, then

$$(3) \quad [f(e^{i\theta})f^*(e^{i\theta})]_{ii} \in L_\infty(\mathbb{T}) \implies f_{ij} \text{ are free of poles on } \mathbb{T}, j = 1, 2, \dots, m,$$

($L_\infty(\mathbb{T})$ stands for the set of bounded functions) since $[f(z)f^*(z)]_{ii} = \sum_{j=1}^m |f_{ij}(z)|^2$ when $|z| = 1$.

$U \in \mathcal{R}^{m \times m}$ is called paraunitary if

$$(4) \quad U(z)U^*(z) = I_m \quad \text{in the domain of } U \text{ and } U^*,$$

where I_m stands for the m -dimensional unit matrix. Note that $U(z)$ is a usual unitary matrix for each $z \in \mathbb{T}$, since $U^*(z) = \overline{U(z)}^T = (U(z))^*$ when $|z| = 1$ and, consequently,

$$(5) \quad \overline{U(z)}^T = U^{-1}(z), \quad z \in \mathbb{T}.$$

Lemma 1. *If $U \in \mathcal{R}^{m \times m}$ is paraunitary and analytic inside \mathbb{T} (its entries are free of poles inside \mathbb{T}), and $U^{-1} \in \mathcal{R}^{m \times m}$ is analytic inside \mathbb{T} as well, then U is a constant unitary matrix.*

Proof. The equation (4) implies that $U_{ij}(z)$, $1 \leq i, j \leq m$, are free of poles on \mathbb{T} (see (3)). Since $U_{ij}(e^{i\theta}) \in L_1^+(\mathbb{T})$ and $L_1^+(\mathbb{T}) \ni U_{ji}^{-1}(e^{i\theta}) = \overline{U_{ij}(e^{i\theta})}$ (see (5)), we have $U_{ij}(e^{i\theta}) \in L_1^+(\mathbb{T}) \cap L_1^-(\mathbb{T})$. Thus $U_{ij}(z)$ is constant for a.a. $z \in \mathbb{T}$, and hence everywhere in the complex plane. \square

Proof of Theorem 1. Since S is nonnegative definite on the unit circle, we have $S^*(z) = S(z)$, $z \in \mathbb{C} \setminus \{0\}$.

Observe that every polynomial matrix-function always has a constant rank in its domain except for a finite number of points. Without loss of generality, we can assume that the $k \times k$ left-upper submatrix of S , denoted by S_{00} , has the full rank k (a.e.) so that S has the block matrix form

$$S(z) = \begin{pmatrix} S_{00}(z) & S_{01}(z) \\ S_{10}(z) & S_{11}(z) \end{pmatrix},$$

where S_{01} , $S_{10} = S_{01}^*$ and S_{11} are matrix-functions of dimensions $k \times (m-k)$, $(m-k) \times k$, and $(m-k) \times (m-k)$, respectively. Since every $k+1$ rows (columns) of $S(z)$ are

linearly dependent, we have

$$(6) \quad S_{10}(z)S_{00}^{-1}(z)S_{01}(z) = S_{11}(z) \quad (\text{a.e.}).$$

Let

$$(7) \quad S_{00}(z) = S_{00}^+(z)S_{00}^-(z) = S_{00}^+(z)(S_{00}^+)^*(z)$$

be the polynomial spectral factorization of S_{00} which exists by virtue of the matrix-valued Fejér-Riesz theorem. Define

$$\sigma_{10}(z) := S_{10}(z)(S_{00}^-(z))^{-1}$$

and let S_0 have the block matrix form

$$S_0(z) = \begin{pmatrix} S_{00}^+(z) \\ \sigma_{10}(z) \end{pmatrix}.$$

Then $S_0^*(z) = [S_{00}^-(z) \quad (S_{00}^+(z))^{-1}S_{01}(z)]$ and, taking (6) into account, one can directly check that

$$(8) \quad S(z) = S_0(z)S_0^*(z).$$

Since S_{00}^+ is a polynomial matrix-function, S_0 is a rational matrix-function, however it might not be analytic inside \mathbb{T} . If s_{ij} is the ij th entry of S_0 with a pole at a inside \mathbb{T} , then we can multiply S_0 by the unitary matrix-function $U(z) = \text{diag}[1, \dots, u(z), \dots, 1]$, where $u(z) = (z - a)/(1 - \bar{a}z)$ is the jj th entry of $U(z)$, so that the ij th entry of the product $S_0(z)U(z)$ will not have a pole at a any longer keeping the factorization (8):

$$(S_0U)(z)(S_0U)^*(z) = S_0(z)S_0^*(z) = S(z).$$

In the same way one can remove every pole of the entries of S_0 at points inside \mathbb{T} . Thus S can be represented as a product

$$(9) \quad S(z) = S_0^+(z)S_0^-(z),$$

where S_0^+ is a rational matrix-function which is analytic inside \mathbb{T} , and $S_0^-(z)$ is its adjoint. Note that $S_0^+(z)$ remains of full rank k for each $z \in \mathbb{T}$ except possibly a finite number of points.

Now, it might happen so that S_0^+ is not of full rank k inside \mathbb{T} everywhere. If $|a| < 1$ and $\text{rank } S_0^+(a) < k$, then there exists a unitary matrix U such that the product $S_0^+(a)U$ has all 0's in the first column. Hence a is a zero of every entry of the first column of the matrix-function $S_0^+(z)U$ and the product $S_1^+(z) := S_0^+(z)U \text{diag}[u(z), 1, \dots, 1]$, where $u(z) = (1 - \bar{a}z)/(z - a)$, remains analytic inside \mathbb{T} . While the factorization (9) remains true replacing S_0^+ and S_0^- by S_1^+ and S_1^- , respectively, the minors of S_1^+ will have less zeros inside \mathbb{T} than the minors of S_0^+ . Thus, continuing this process if necessary, we get the factorization

$$(10) \quad S(z) = S^+(z)S^-(z),$$

where S^+ is a rational matrix-function which is analytic and of full rank k inside \mathbb{T} .

Now let us show that S^+ is in fact a polynomial matrix-function of order N . It suffices to show that $z^N S^-(z)$ is analytic inside \mathbb{T} . Indeed, since S^+ does not have poles on \mathbb{T} (see (10) and (3)), $z^N S^-(z)$ should be an analytic (on the whole \mathbb{C}) rational matrix-function in this case, and therefore a polynomial.

It follows from (10) that

$$(11) \quad z^N S^-(z) = ((S^+(z))^T S^+(z))^{-1} \cdot (S^+(z))^T \cdot z^N S(z)$$

and $z^N S^-(z)$ is analytic inside \mathbb{T} since each of the three factors on the right-hand side of (11) is such.

To complete the proof of the theorem, it remains to show that the factorization (2) is unique, i.e. if

$$S(z) = S_1^+(z) S_1^-(z)$$

where S_1^+ is a $m \times k$ polynomial matrix-function which has the full rank k inside \mathbb{T} , then

$$S_1^+(z) = S^+(z) U$$

for some $k \times k$ (constant) unitary matrix U .

Since $S^+(z)$ is of the full rank k for each $z \in \mathbb{C}$ except for some finite number of singular points, there exists a matrix-function $U(z)$ such that

$$(12) \quad S_1^+(z) = S^+(z) U(z)$$

Thus $U(z)$ can be determined by the equation

$$U(z) = ((S^+)^T(z) S^+(z))^{-1} (S^+)^T(z) S_1^+(z)$$

as a rational function in \mathbb{C} . Note that $U(z)$ is analytic inside \mathbb{T} , and since S^+ and S_1^+ participate symmetrically in the theorem, $U^{-1}(z)$ is analytic inside \mathbb{T} as well.

Due to Lemma 1, it remains to show that $U \in \mathcal{R}^{k \times k}$ is a paraunitary matrix-function. From the equation (12), one can determine $U(z)$ as

$$U(z) = (S^-(z) S^+(z))^{-1} S^-(z) S_1^+(z)$$

and, consequently,

$$U^*(z) = S_1^-(z) S^+(z) (S^-(z) S^+(z))^{-1}.$$

Hence

$$\begin{aligned} U(z) U^*(z) &= (S^-(z) S^+(z))^{-1} S^-(z) S_1^+(z) \cdot S_1^-(z) S^+(z) (S^-(z) S^+(z))^{-1} = \\ &= (S^-(z) S^+(z))^{-1} S^-(z) S^+(z) S^-(z) S^+(z) (S^-(z) S^+(z))^{-1} = I_k. \end{aligned}$$

The proof of the theorem is complete.

Remark. As one can observe, the above proof of the existence of S^+ is constructive. There are several classical algorithms to perform the factorization (7) numerically in the full rank case (a new efficient algorithm of such type is proposed in [5]). Further using the steps described in the proof, one can compute S^+ numerically.

Our next theorem illustrates one of the applications of Theorem 1 in some areas of signal processing. Namely, $m \times m$ paraunitary matrix-functions

$$(13) \quad U(z) = \sum_{n=0}^N \rho_n z^n = [u_{ij}(z)]_{i,j=\overline{1,m}}, \quad \rho_n \in \mathbb{C}^{m \times m},$$

defined by (4) play an important role in the theory of wavelets and multirate filter banks [8] where they are known under different names, for example, lossless systems [11], perfect reconstruction m -filters [7], paraunitary m -channel filters [10], and so on.

The positive integers m and N are called the *size* and the *length* of U , respectively. Sometimes, the first row of a matrix-function U is called the *low-pass filter*, and the remaining rows are called the *high-pass filters*. Theorem 2 allows us to find the set of matching high-pass filters to each low-pass filter. First we give a simple proof of the following lemma which provides additional information about structures of paraunitary matrix-polynomials.

Lemma 2. (cf. [8, Lemma 4.13]) *Let (13) be a paraunitary matrix-polynomial of length N ($\rho_N \neq \mathbf{0}$). Then*

$$(14) \quad \det U(z) = c \cdot z^k, \text{ where } |c| = 1, \text{ and } k \geq N.$$

Proof. Since $\det U(z) \cdot \det U^*(z) = 1$ and $\det U(z)$ is a polynomial, it follows that $\det U(z) = cz^k$ for some nonnegative integer k . We have

$$(15) \quad \sum_{n=0}^N \rho_n^* z^{-n} = U^*(z) = U^{-1}(z) = \frac{1}{\det U(z)} (\text{Cof } U(z))^T = cz^{-k} (\text{Cof } U(z))^T.$$

Therefore $k \geq N$, since $\text{Cof } U(z)$ is a polynomial matrix-function and ρ_N^* is not the zero matrix. \square

Remark. The positive integer k in (14) is called the *degree* of U . Generically, a paraunitary matrix-polynomial U of length N has the same degree N , although in some specific cases the degree is more than N .

The following theorem was first established in [3] by a different method, however the presented approach gives a new insight to the problem.

Theorem 2. *For any polynomial vector-function*

$$(16) \quad U_1(z) = [u_{11}(z), u_{12}(z), \dots, u_{1m}(z)],$$

$u_{1j}(z) = \sum_{n=0}^N \alpha_{jn} z^n$, $j = 1, 2, \dots, m$, of length N ($\sum_{j=1}^m |\alpha_{jn}| > 0$) which is of unit norm on \mathbb{T}

$$(17) \quad \|U_1(z)\|_{\mathbb{C}^m}^2 = \sum_{j=1}^m |u_{1j}(z)|^2 = 1, \quad z \in \mathbb{T},$$

there exists a unique (up to a constant left multiplier of the block matrix form $\begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix}$, where U is a $(m-1) \times (m-1)$ unitary matrix) paraunitary matrix-function $U(z)$ (of size m and length N), with determinant cz^N , $|c| = 1$, whose first row is equal to (16).

Lemma 3. *Let $\mathbf{v} = (v_1, v_2, \dots, v_m)^T \in \mathbb{C}^m$ be a vector of unit norm, $\|\mathbf{v}\|^2 = \mathbf{v}^* \mathbf{v} = \sum_{j=1}^m |v_j|^2 = 1$. Then $I_m - \mathbf{v} \mathbf{v}^*$ is a nonnegative definite matrix,*

$$(18) \quad I_m - \mathbf{v} \mathbf{v}^* \geq 0,$$

and

$$(19) \quad \text{rank}(I_m - \mathbf{v} \mathbf{v}^*) = m - 1.$$

Proof. For each column vector $\mathbf{x} \in \mathbb{C}^m$, we have

$$\mathbf{x}^*(I_m - \mathbf{v}\mathbf{v}^*)\mathbf{x} = \|\mathbf{x}\|^2 - |\mathbf{x}^*\mathbf{v}|^2 \geq \|\mathbf{x}\|^2 - \|\mathbf{x}^*\|^2\|\mathbf{v}\|^2 = \|\mathbf{x}\|^2 - \|\mathbf{x}^*\|^2 = 0.$$

Hence (18) holds and $\mathbf{x}^*(I_m - \mathbf{v}\mathbf{v}^*)\mathbf{x} = 0$ if and only if $\mathbf{x} = \alpha\mathbf{v}$ for some $\alpha \in \mathbb{C}$. Thus (19) holds as well. \square

Proof of Theorem 2. Due to Lemma 3 and the property (17), the matrix-function

$$(20) \quad S(z) = I_m - U_1^T(z)(U_1^T)^*(z)$$

is positive definite and of rank $m - 1$ for each $z \in \mathbb{T}$. (Note that the order of S is less than or equal to N .) Hence, by virtue of Theorem 1, there exists an $m \times (m - 1)$ matrix-function $S^+(z)$ of full rank $m - 1$, for each z inside \mathbb{T} , such that (2) holds. Consequently,

$$\begin{bmatrix} U_1^T(z) & S^+(z) \end{bmatrix} \begin{bmatrix} (U_1^T)^*(z) \\ S^-(z) \end{bmatrix} = I_m$$

and

$$U(z) = \begin{bmatrix} U_1(z) \\ (S^+)^T(z) \end{bmatrix}$$

is the paraunitary matrix-function we wanted to find. Indeed, clearly $U(z)$ is of size m and length N , and we show that

$$(21) \quad \det U(z) = c \cdot z^N, \quad |c| = 1.$$

Due to Lemma 2, $\det U(z) = cz^k$, $|c| = 1$, for some positive integer $k \geq N$. Hence (see (15))

$$(22) \quad \sum_{n=0}^N \overline{\alpha_{jn}} z^{-n} = u_{1j}^*(z) = c \cdot z^{-k} \cdot \text{cof}(u_{1j}(z)), \quad j = 1, 2, \dots, m.$$

Since $S^+(0)$ is of rank $m - 1$, then $\text{cof}(u_{1j}(0)) \neq 0$ for at least one $j \in \{1, 2, \dots, m\}$ so that the first coefficient of the polynomial $\text{cof}(u_{1j}(z))$ differs from 0 for at least one j . Thus it follows from (22) that $k \leq N$ and hence $k = N$, which yields (21). The desired $U(z)$ is found and let us show its uniqueness.

Assume now that $U(z)$ is any $m \times m$ paraunitary polynomial matrix-function, with the first row (16), which satisfies (21), and let $U_{m-1}(z)$ be the $(m - 1) \times m$ matrix-polynomial which is formed by deleting the first row in $U(z)$. It is obvious that $U_{m-1}^T(z)$ is an $m \times (m - 1)$ polynomial spectral factor of (20) so that, by virtue of Theorem 1, we get $U_{m-1}^T(z) = S^+(z)U \iff U(z) = \begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix} \begin{pmatrix} U_1(z) \\ (S^+)^T(z) \end{pmatrix}$ immediately after we establish that $U_{m-1}(z)$ is of full rank $m - 1$ for each z inside \mathbb{T} . But $\text{rank } U_{m-1}(z) = m - 1$ for any $z \neq 0$ since (21) implies that $\text{rank } U(z) = m$, $z \neq 0$, and $\sum_{n=0}^N \overline{\alpha_{jn}} z^{-n} = u_{1j}^*(z) = c z^{-N} \text{cof}(u_{1j}(z))$ (see (22)), $\alpha_{jN} \neq 0$, implies that $\text{cof}(u_{1j}(0)) \neq 0$, which means that $\text{rank } U_{m-1}(0) = m - 1$.

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